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Extreme Values in FGM Random Sequences

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We consider the multivariate Farlie–Gumbel–Morgenstern class of distributions and discuss their properties with respect to the extreme values. This class was used to consider dependence in multivariate distributions and their ordering. We show that the extreme values of these distributions behave as if no dependence would exist between its components. © 1999 Academic Press

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1. INTRODUCTION

In this paper we consider extremes of the Farlie–Gumbel–Morgenstern class of multivariate distributions which was used for the construction of multivariate distributions (see Conway, 1983). It was proposed by Morgenstern (1956), extended by Farlie (1960), and is now known as the Farlie–Gumbel–Morgenstern (FGM) class of distributions. These distributions have a simple natural form with given univariate marginals. This class was further generalized to include distributions with a stronger correlation structure, see, e.g., Johnson and Kotz (1975, 1977) and Cambanis (1977). For a recent discussion of this family of distributions see Lin (1987), Kotz and Seeger (1993), and Cambanis (1993).

Cambanis (1993) considered the question whether this class of multivariate FGM distributions might be the family of finite dimensional distributions of a stationary random sequence or a stochastic process with continuous time. He showed that this class reveals a dependence structure which is rather restricted. It does not include for instance complete dependence nor strong dependence, in general. He noted in addition that "these simple models of dependence may be inappropriate for sampled time or spatial processes."

Our motivation for this short note consists in analyzing the extreme value behavior of such stationary or nonstationary random sequences,

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where we deal with univariate as well as multivariate sequences. Because of the restricted dependence structure the behavior of the extreme values is mostly not influenced by the dependence structure, asymptotically.

A FGM distribution H in $\mathbb{R}^n$, for $n \geqslant 1$, is defined with respect to given univariate distributions F_i, $i \leqslant n$, by

$$H(x_1, ..., x_n) = \prod_{i=1}^{n} F_i(x_i) \left\{1 + \sum_{1 \leqslant j < k \leqslant n} a(j, k)\, \bar{F}_j(x_j)\, \bar{F}_k(x_k)\right\}$$

for all vectors $\mathbf{x} = (x_1, ..., x_n) \in \mathbb{R}^n$ where the $n(n-1)/2$ terms $a(j, k)$ are suitable constants, such that H is a distribution function. The univariate marginals of H are the given F_i. The constants $a(j, k)$ are admissible if the 2^n inequalities

$$1 + \sum_{1 \leqslant j < k \leqslant n} \varepsilon_j \varepsilon_k a(j, k) \geqslant 0$$

hold, for all $\varepsilon_j = -M_j$ or $1 - m_j$, where M_j and m_j are the supremum and the infimum of the set

$$\{F_j(x), -\infty < x < \infty\} \backslash \{0, 1\}.$$

If F_j is absolutely continuous, then $M_j = 1$ and $m_j = 0$, hence $\varepsilon_j = \pm 1$. These inequalities imply that the coefficients are bounded, for instance by $|a(j, k)| \leqslant 1/[\min\{M_k, M_j, (1-m_j), (1-m_k)\}]^2$, which follows immediately using the bivariate distributions. We assume that the distributions F_i are nondegenerated with $\inf_{j \geqslant 1} M_j > 0$ and $\sup_{j \geqslant 1} m_j < 1$. Note that the multivariate distributions are determined by the bivariate marginals (by the terms $a(j, k)$ and the univariate F_i) and that their k-dimensional marginals are also of the same type. More general FGM distributions were proposed and analyzed in the above-mentioned papers. However, we note in Section 3 that the behavior of the extreme values of these more general FGM random sequences is not different from the one which is analyzed in the following.

A FGM random sequence $\{X_i, i \geqslant 1\}$ is now defined by the univariate marginals $F_i \sim X_i$, $i \geqslant 1$, and a symmetric function $a(\cdot)$ (that means $a(j, k) = a(k, j)$) such that the joint distribution of $X_{i_1}, ..., X_{i_n}$ is given by the FGM distribution

$$H_{i_1, ..., i_n}(\mathbf{x}) = \prod_{h=1}^{n} F_{i_h}(x_i) \left\{1 + \sum_{1 \leqslant j < k \leqslant n} a(i_j, i_k)\, \bar{F}_{i_j}(x_j)\, \bar{F}_{i_k}(x_k)\right\},$$

where $\mathbf{x} = (x_1, ..., x_n)$. The function $a(\cdot)$ is admissible if for every $n \geqslant 1$ and $\{i_1, ..., i_n\}$ the inequalities

$$1 + \sum_{1 \leqslant j < k \leqslant n} \varepsilon_{i_j} \varepsilon_{i_k} a(i_j, i_k) \geqslant 0$$

hold for all ε_{i_j}.

The FGM sequence is stationary iff the univariate marginals are all equal,

$$F_i(\cdot) = F_1(\cdot), \qquad i > 1,$$

and the function $a(j, k)$ depends on j, k only through their difference:

$$a(j, k) = a(j - k) \qquad \text{for all} \quad j \neq k.$$

In this case we use the same notation for the function $a(\cdot)$ having one argument only. In the following the function $a(\cdot)$ plays the same role in the construction of a FGM distribution always even if $a(\cdot)$ has four arguments.

In the same way we can introduce an independent sequence of random vectors $\mathbf{X}_i$, $i \geqslant 1$, where the distribution H_i of $\mathbf{X}_i$ is a d-dimensional FGM distribution with marginals $F_{i,j}$, $j \leqslant d$, and coefficients $a_i(j, k)$, $j, k \leqslant d$. Obviously, if the $H_i \equiv H_1$, the random vectors $\mathbf{X}_i$ are identically distributed which happens if $a_i(j, k) = a_1(j, k)$ and $F_{i,j} = F_{1,j}$ for all $i > 1$, $j, k \leqslant d$.

A further extension is possible by defining a multivariate stationary or nonstationary random sequence of d-dimensional random vectors $\mathbf{X}_i$. Using the idea of the construction of the FGM distributions we might define for instance

$$H_{i_1, ..., i_k}(\mathbf{x}_1, ..., \mathbf{x}_k) = \prod_{h=1}^{k} H_{i_h}(\mathbf{x}_h) \left\{ 1 + \sum_{1 \leqslant h < h' \leqslant k} a(i_h, i_{h'}) \bar{H}_{i_h}(\mathbf{x}_h) \bar{H}_{i_{h'}}(\mathbf{x}_{h'}) \right\}$$

for suitable coefficients $a(\cdot, \cdot)$. However, this distribution in $\mathbb{R}^{kd}$ is not of the FGM type. Therefore we define the FGM random sequence $\mathbf{X}_i$, $i \geqslant 1$, such that for any $k \geqslant 1$ and $i_1 < \cdots < i_k$ the kd-dimensional distribution of $X_{i_1,1}, ..., X_{i_1,d}, ..., X_{i_k,d}$ is a FGM distribution with respect to a function $a(\cdot)$,

$$\begin{aligned} H_{i_1, ..., i_k}&(\mathbf{x}_1, ..., \mathbf{x}_k) \\ &= \prod_{h=1}^{k} \prod_{j=1}^{d} F_{i_h, j}(\mathbf{x}_h) \\ &\quad \times \left\{ 1 + \sum_{1 \leqslant h \leqslant h' \leqslant k} \sum_{\substack{1 \leqslant l, l' \leqslant d, \\ (h, l) < (h', l')}} a(i_h, i_{h'}; l, l') \bar{F}_{i_h, l}(x_{hl}) \bar{F}_{i_{h'}, l'}(x_{h'l'}) \right\} \end{aligned}$$

for admissible coefficients $a(h, h'; l, l')$, $h, h' \geqslant 1$, $l, l' \leqslant d$, where $F_{i,l}(x) = P\{X_{il} \leqslant x\}$. Note that $(h, l) < (h', l')$ means that either $h < h'$ or $h = h'$ and $l < l'$.

We consider the partial maxima

$$M_n = \max_{i \leqslant n} X_i, \qquad n \geqslant 1,$$

in the univariate case, and the vector of componentwise partial maxima

$$\mathbf{M}_n = (M_{n1}, ..., M_{nd}) = (\max_{i \leqslant n} X_{i1}, ..., \max_{i \leqslant n} X_{id})$$

in the multivariate case.

We present in the next section some results on the limiting distribution of the maxima for the univariate case and in Section 3 for the multivariate case.

2. UNIVARIATE CASE

Let $\{X_i, i \geqslant 1\}$ be a sequence of random variables X_i where their joint finite-dimensional distributions are FGM. The limiting distribution of the maxima is derived with respect to some suitable normalization $u_n(x)$. We want to consider the approximation of $P\{M_n \leqslant u_n(x)\}$. In this class the dependence between the random variables X_i is not very strong. This means that the approximation

$$P\{M_n \leqslant u_n(x)\} \approx \prod_{i \leqslant n} P\{X_i \leqslant u_n(x)\} = \prod_{i \leqslant n} F_i(u_n(x))$$

holds. Only in a few cases is this approximation not suitable.

In the general case of nonidentically distributed X_i the following u.a.n. (uniform asymptotic negligibility) condition is essential for general results. Suppose that for the normalization $u_n(x)$

$$p_{n,\max} := \sup_{i \leqslant n} \bar{F}_i(u_n(x)) \to 0$$

as $n \to \infty$, for the set of x with $\liminf_{n \to \infty} \prod_{i \leqslant n} F_i(u_n(x)) > 0$. For some sequences F_i and normalizations $u_n(x)$, the limiting distribution G of the maxima M_n exists as

$$\lim_{n \to \infty} \prod_{i \leqslant n} F_i(u_n(x)) = G(x). \tag{1}$$

Note that G in (1) is not necessarily an extreme value distribution even under a linear normalization (see Galambos, 1987; Hüsler, 1989a; Falk *et al.*, 1994).

In some cases, e.g., nonstationary sequences (see, e.g., Hüsler, 1983) it is reasonable to replace the normalization $u_n(x)$ by a boundary $\{u_{ni},\ i \leqslant n,\ n \geqslant 1\}$ which is nonconstant in i for fixed n; u_{ni} may depend on x also. The more general u.a.n. condition is now

$$p_{n,\,\max} = \sup_{i \leqslant n} \bar{F}_i(u_{ni}) \to 0, \qquad \text{as} \quad n \to \infty. \tag{2}$$

A general theory of extreme values is established for random sequences satisfying some mixing conditions. Usually a weak distributional mixing condition $D = D(u_{ni})$ is supposed with respect to some more general norming or boundary values $\{u_{ni}, i \leqslant n, n \geqslant 1\}$ (see Leadbetter *et al.*, 1983; Falk *et al.*, 1994). For each n and m let $\alpha_{n,\,m}$ be such that for any $1 \leqslant i_1 < i_2 < \cdots < i_p < j_1 < \cdots < j_q \leqslant n$ with $j_1 - i_p \geqslant m$,

$$|P(X_l \leqslant u_{nl}, l \in I \cup J) - P(X_l \leqslant u_{nl}, l \in I)\ P(X_l \leqslant u_{nl}, l \in J)| \leqslant \alpha_{n,\,m}$$

with $I = \{i_k, k \leqslant p\}$ and $J = \{j_k, k \leqslant q\}$. Condition D is said to hold if

$$\alpha_{n,\,m_n} \to 0, \qquad \text{as} \quad n \to \infty$$

for some sequence $m_n \to \infty$ (as $n \to \infty$) with $m_n p_{n,\,\max} \to 0$.

The condition D holds for a FGM sequence if the coefficients $a(j, k)$ satisfy the simple sufficient condition

$$\sup_{j-k>n} |a(j, k)| \to 0, \qquad n \to \infty \tag{3}$$

We do not know whether this condition holds for all FGM sequences. Cambanis (1993) has shown that the dependence structure of FGM stationary sequences is rather restricted. For instance, (i) m-dependent FGM sequences exist only as independent sequences (proved for $m \leqslant 3$); (ii) equal or constant dependence is not possible, again only independence is possible; (iii) a positive geometric decay of the coefficients is also not possible. Under condition (3) the limiting distribution of the maxima is derived by the limiting behavior of $\prod_{i \leqslant n} F_i(u_n(x))$.

Lemma 1. *Assume that the FGM sequence X_i is u.a.n. and that* (3) *holds. Then Condition* D *holds for any boundary values u_{ni} such that*

$$\limsup_{n \to \infty} \sum_{i \leqslant n} \bar{F}_i(u_{ni}) < \infty. \tag{4}$$

Proof. Note that for any $I \subset \{1, ..., n\}$,

$$P(X_l \leqslant u_{nl}, l \in I) = \prod_{l \in I} F_l(u_{nl}) \left(1 + \sum_{l < l' \in I} a(l, l')\, \bar{F}_l(u_{nl})\, \bar{F}_{l'}(u_{nl'})\right).$$

Using (3) the absolute value of the double sum can be bounded by

$$\sum_{l < l' \in I,\, l' - l \leqslant l_0} |a(l, l')|\, \bar{F}_l(u_{nl})\, p_{n,\max} + \sum_{l < l' \in I,\, l' - l > l_0} \varepsilon \bar{F}_l(u_{nl})\, \bar{F}_{l'}(u_{nl'})$$

$$= O\left(l_0\, p_{n,\max} \sum_{l \leqslant n} \bar{F}_l(u_{nl})\right) + O\left(\varepsilon \left(\sum_{l \leqslant n} \bar{F}_l(u_{nl})\right)^2\right)$$

for any $\varepsilon > 0$ and with suitable l_0 such that $|a(l, l')| < \varepsilon$ for $l' - l > l_0$. Using (4) this double sum converges to 0 as $n \to \infty$. This holds uniformly for any $I \subset \{1, ..., n\}$. Further it implies

$$|P(X_l \leqslant u_{nl}, l \in I \cup J) - P(X_l \leqslant u_{nl}, l \in I)\, P(X_l \leqslant u_{nl}, l \in J)|$$

$$\leqslant \prod_{l \in I \cup J} F_l(u_{nl})(1 + o(1) - (1 + o(1))^2) = o(1)$$

for any m, uniformly for all I, J. ∎

Therefore a general proposition holds for a nonstationary sequence satisfying (3).

Proposition 2. *Let* $\{X_i, i \geqslant 1\}$ *be an* u.a.n. FGM *random sequence such that* (3) *and* (4) *hold with respect to some normalization* $u_n(x)$. *Then*

$$P\{M_n \leqslant u_n(x)\} - \prod_{i \leqslant n} F_i(u_n(x)) \to 0 \tag{5}$$

as $n \to \infty$. *If in addition* (1) *holds, then*

$$P\{M_n \leqslant u_n(x)\} \xrightarrow{d} G(x)$$

as $n \to \infty$.

Proof. Note that by the FGM structure of the random sequence we have

$$P\{M_n \leqslant u_n(x)\} = \prod_{i \leqslant n} F_i(u_n(x)) \left\{1 + \sum_{1 \leqslant j < k \leqslant n} a(j, k)\, \bar{F}_j(u_n(x))\, \bar{F}_k(u_n(x))\right\}$$

$$= \prod_{i \leqslant n} F_i(u_n(x)) \{1 + o(1)\}$$

iff

$$\sum_{1 \leqslant j < k \leqslant n} a(j, k)\, \bar{F}_j(u_n(x))\, \bar{F}_k(u_n(x)) = o(1). \tag{6}$$

Equation (6) is implied by the assumptions as in Lemma 1. ∎

The proof shows also the converse statement that (5) implies (6). In the stationary case (6) means that $\sum_{j<k\leqslant n} a(j,k) = o(n^2)$, since by (4) $\limsup_n n\bar{F}(u_n(x)) < \infty$. In the Appendix we show that condition (6) holds always in the stationary case. Also it is proved that a slightly stronger condition (6′) implies D for general nonstationary sequences. We believe that (6) implies D in most cases; we did not find a counterexample. Note also that the two conditions are not the same in structure and that D implies quite strong extreme value results.

3. MULTIVARIATE CASE

Let $\{\mathbf{X}_i, i \geqslant 1\}$ be a sequence of independent random vectors $\mathbf{X}_i$ in $\mathbb{R}^d$ where their distributions H_i are FGM. In the following the algebraic operations are meant componentwise.

LEMMA 3. *Let $\{\mathbf{X}_i, i \geqslant 1\}$ be a sequence of independent r.v.'s in $\mathbb{R}^d$ with FGM distributions H_i. Let $\{\mathbf{u}_{ni}, i \leqslant n, n \geqslant 1\}$ be a boundary sequence with $\mathbf{u}_{ni} = (u_{ni,1}, ..., u_{ni,d})$ such that* (4) *and the u.a.n. condition hold for each component $j \leqslant d$. Then*

$$P\{\mathbf{X}_i \leqslant \mathbf{u}_{ni}, i \leqslant n\} - \prod_{i \leqslant n} \prod_{j \leqslant d} F_{i,j}(u_{ni,j}) \to 0, \qquad n \to \infty.$$

Proof. By the independence of the $\mathbf{X}_i$'s

$$\begin{aligned} P\{\mathbf{X}_i \leqslant \mathbf{u}_{ni}, i \leqslant n\} &= \prod_{i \leqslant n} H_i(\mathbf{u}_{ni}) \\ &= \prod_{i \leqslant n} \left[\prod_{j \leqslant d} F_{i,j}(u_{ni,j}) \left(1 + \sum_{1 \leqslant j < k \leqslant d} a_i(j,k)\, \bar{F}_{i,j}(u_{ni,j})\, \bar{F}_{i,k}(u_{ni,k}) \right) \right]. \end{aligned}$$

Since $p_{n,\max} = \sup_{i \leqslant n} \sup_{j \leqslant d} \bar{F}_{i,j}(u_{ni,j}) \to 0$, the double sum converges to 0 as $n \to \infty$ and also

$$\sum_{i \leqslant n} \sum_{1 \leqslant j < k \leqslant d} a_i(j, k)\, \bar{F}_{i,j}(u_{ni,j})\, \bar{F}_{i,k}(u_{ni,k})$$

$$= O\left(\sup_{j \leqslant d}\left(\sum_{i \leqslant n} \bar{F}_{i,j}(u_{ni,j})\right) p_{n,\max}\right) = o(1).$$

This implies the statement. ∎

In the case of i.i.d. r.v.'s, $\mathbf{X}_i$ a limiting distribution of $\mathbf{M}_n$ exists if every marginal F_j, $j \leqslant d$, of $X_{i,j}$ belongs to some domain of attraction of an extreme value distribution G_j, i.e., there exist norming values a_{nj} (>0) and b_{nj} such that

$$P\{M_{nj} \leqslant a_{nj}x + b_{nj}\} = F_j^n(a_{nj}x + b_{nj}) \xrightarrow{d} G_j(x) \tag{7}$$

as $n \to \infty$. We use the notation $\mathbf{a}_n = (a_{n1}, ..., a_{nd})$, $\mathbf{b}_n = (b_{n1}, ..., b_{nd})$, and $\mathbf{u}_n(\mathbf{x}) = (u_{n1}(x_1), ..., u_{nd}(x_d))$ where $u_{nj}(x_j) = a_{nj}x_j + b_{nj}$.

Proposition 4. *If the i.i.d.* FGM *random sequence* $\{\mathbf{X}_i, i \geqslant 1\}$ *is such that* (7) *holds, then*

$$P\{\mathbf{M}_n \leqslant \mathbf{a}_n\mathbf{x} + \mathbf{b}_n\} \xrightarrow{d} \prod_{j \leqslant d} G_j(x_j)$$

as $n \to \infty$.

Proof. Equation (7) implies (4) and the u.a.n. condition for each component. Hence the statement follows by Lemma 3. ∎

Furthermore, we derive easily for i.i.d. sequences

$$\begin{aligned} &P\{\mathbf{M}_n \leqslant \mathbf{a}_n\mathbf{x} + \mathbf{b}_n\} \\ &\quad = \prod_{j=1}^{d} F_j^n(u_{nj}(x_j)) \left\{1 + \sum_{1 \leqslant k < l \leqslant d} a(k, l)\, \bar{F}_k(u_{nk}(x_k))\, \bar{F}_l(u_{nl}(x_l))\right\}^n \\ &\quad = \prod_{j=1}^{d} F_j^n(u_{nj}(x_j)) \{1 + O(1/n^2)\}^n \\ &\quad \xrightarrow{d} \prod_{j=1}^{d} G_j(x_j) = G(\mathbf{x}), \end{aligned}$$

since $\bar{F}_j(u_{nj}(x_j)) = O(1/n)$ and $\sum_{1 \leqslant k < l \leqslant d} a(k, l) = O(1)$. This implies the general bound of the convergence rate

$$\left| P\{\mathbf{M}_n \leqslant \mathbf{u}_n(\mathbf{x})\} - \prod_{j \leqslant d} F_j^n(u_n(x_j)) \right| \leqslant \prod_{j \leqslant d} F_j^n(u_n(x_j)) \, |[1 + O(1/n^2)]^n - 1| = O(1/n)$$

uniformly for $\mathbf{x}$. Hence the speed of convergence of $P\{\mathbf{M}_n \leqslant \mathbf{u}_n(\mathbf{x})\}$ to the limiting distribution depends on the speed of convergence of each component plus this $O(1/n)$ term. The dependence of the components M_{nj} is asymptotically vanishing at a rather fast speed.

This result can be also extended to the case of independent but non-identically distributed random vectors $\mathbf{X}_i$. We need to assume that the distribution of the component M_{nj} converges with suitable (linear) normalization $u_{nj}(x)$. Also in this multivariate case the limiting distribution is in general not an extreme value distribution (see, e.g., Hüsler, 1989a, b; or Falk *et al.*, 1994).

PROPOSITION 5. *Assume that the i.non-i.d.* FGM *random sequence* $\{\mathbf{X}_i, i \geqslant 1\}$ *is such that for each component* j, $j \leqslant d$, *the* FGM *sequence* $\{X_{i,j}, i \geqslant 1\}$ *is a u.a.n. and*

$$P\{M_{nj} \leqslant u_{nj}(x)\} \xrightarrow{d} G_j(x)$$

as $n \to \infty$, *with suitable normalizations* $u_{nj}(x)$. *Then*

$$P\{\mathbf{M}_n \leqslant \mathbf{u}_n(\mathbf{x})\} \xrightarrow{d} \prod_{j \leqslant d} G_j(x_j)$$

as $n \to \infty$.

The proof follows the same lines as in the proof of Proposition 2 and Lemma 3. Finally we consider nonstationary multivariate FGM random sequences (introduced in Section 1) in the following proposition which is proved along the same lines. Instead of (3) we use

$$\sup_{h' - h > m} \sup_{1 \leqslant l, l' \leqslant d} |a(h, h'; l, l')| \to 0, \qquad n \to \infty. \tag{8}$$

PROPOSITION 6. *Assume that the* FGM *random sequence* $\{\mathbf{X}_i, i \geqslant 1\}$ *is such that* (8) *holds and that for each component* j, $j \leqslant d$, *the* FGM *sequence* $\{X_{i,j}, i \geqslant 1\}$ *is u.a.n. with respect to the normalization* $u_{nj}(x)$. *Then*

$$P\{M_{nj} \leqslant u_{nj}(x)\} - \prod_{i \leqslant n} \prod_{j \leqslant d} F_{i,j}(u_{nj}(x_j)) \to 0, \qquad n \to \infty.$$

If in addition (1) *holds for each component* $j \leqslant d$, *i.e.*,

$$\prod_{i \leqslant n} F_{i,j}(u_{nj}(x)) \xrightarrow{d} G_j(x)$$

as $n \to \infty$, *then*

$$P\{\mathbf{M}_n \leqslant \mathbf{u}_n(\mathbf{x})\} \xrightarrow{d} \prod_{j \leqslant d} G_j(x_j)$$

as $n \to \infty$.

Remark. Johnson and Kotz (1975, 1977) introduced a more general class of FGM distribution with a stronger dependence between the components. Such a FGM distribution is given by

$$H(\mathbf{x}) = \prod_{j=1}^{d} F_j(x_j) \left\{1 + \sum_{g=2}^{d} \sum_{1 \leqslant j_1 < j_2 < \cdots < j_g \leqslant d} a(j_1, j_2, ..., j_g) \prod_{h=1}^{g} \bar{F}_{j_h}(x_h)\right\},$$

where again the constants $a(\cdot)$ fulfill some conditions. But the bivariate marginals of H are of the same type as the FGM dealt with in the beginning. Therefore Lemma 3 implies that the bivariate dependence is asymptotically negligible for the events related to extremes. Since the bivariate independence of the components of $\mathbf{M}_n$ implies the multivariate independence of all components (Hüsler, 1989a; cf. Hüsler, 1994, Theorem 3.4), we get the same results as Proposition 4, 5, and 6 (by assuming obviously instead of (8) a suitably adapted condition) for the more generale FGM distributions.

APPENDIX

(1) We prove now that a stationary FGM sequence satisfies (6) which means

$$\sum_{i<j} a(j, k) = o(n^2)$$

since by (4) $\bar{F}(u_n(x)) = O(n^{-1})$. The constants $a(j, k)$ are admissible if

$$1 + \sum_{1 \leqslant j < k \leqslant n} \varepsilon_j \varepsilon_k a(j, k) \geqslant 0 \tag{9}$$

for any $\varepsilon_j = -M_j$ or $1 - m_j$. In the stationary case we have $M_j = M$ and $m_j = m$ for all $j \geqslant 1$. In our case of extreme value distributions, we have

$M=1$, which is not used in the following proof. For any choice of $\varepsilon=(\varepsilon_1, ..., \varepsilon_n) \in \mathbb{R}^n$, we define the sets

$$J^{-}=\{(j,k): \varepsilon_j<0, \varepsilon_k<0\},$$

$$J^{+-}=\{(j,k): \varepsilon_j>0, \varepsilon_k<0\},$$

$$J^{-+}=\{(j,k): \varepsilon_j<0, \varepsilon_k>0\},$$

$$J^{++}=\{(j,k): \varepsilon_j>0, \varepsilon_k>0\}.$$

Then (9) can be written as

$$\sum_{j<k,\,(j,k)\in J^{-+}\cup J^{+-}} M(1-m)\,a(j,k)-\sum_{j<k,\,(j,k)\in J^{++}} (1-m)^2\,a(j,k)$$
$$-\sum_{j<k,\,(j,k)\in J^{--}} M^2 a(j,k)\leqslant 1.$$

We sum on all these equations when $\varepsilon \in \mathbb{R}^n$ such that $|\{j: \varepsilon_j<0\}|=n^*$. Some of these might be the same, but this does not matter. We get

$$S=S_1-S_2-S_3\leqslant\binom{n}{n^*}.$$

We consider now an element $a(j,k)$ with $1\leqslant j<k\leqslant n$. This element appears

$$2\binom{n-2}{n^*-1}$$

times in the first sum S_1, $\binom{n-2}{n^*}$ in S_2 and $\binom{n-2}{n^*-2}$ in S_3. Hence we have for the sum S on the $\binom{n}{n^*}$ choices of ε

$$\sum_{1\leqslant j<k\leqslant n} a(j,k)\,c\leqslant\binom{n}{n^*},$$

where

$$c:=2M(1-m)\binom{n-2}{n^*-1}-(1-m)^2\binom{n-2}{n^*}-M^2\binom{n-2}{n^*-2}.$$

Simplifying we get

$$c=\frac{2M(1-m)(n-2)!}{(n^*-1)!\,(n-n^*-1)!}-\frac{(1-m)^2\,(n-2)!}{n^*!\,(n-n^*-2)!}-\frac{M^2(n-2)!}{(n^*-2)!\,(n-n^*)!}$$

$$=\binom{n-2}{n^*-1}\left\{\frac{\left[\begin{array}{c}2M(1-m)\,n^*(n-n^*)-(1-m)^2\,(n-n^*-1)(n-n^*)\\ -M^2n^*(n^*-1)\end{array}\right]}{n^*(n-n^*)}\right\}$$

$$=\binom{n-2}{n^*-1}\left\{\frac{M^2n^*+(1-m)^2\,(n-n^*)-((1-m)(n-n^*)-Mn^*)^2}{n^*(n-n^*)}\right\}.$$

Choosing $n^*=\lfloor(1-m)/(1-m+M)\,n\rfloor\sim\lambda n$, with $0<\lambda<1$, the constant c is positive, since

$$((1-m)(n-n^*)-Mn^*)^2\leqslant 4$$

and hence

$$c\geqslant\binom{n-2}{n^*-1}\frac{M^2n^*+(1-m)^2\,(n-n^*)-4}{n^*(n-n^*)}$$

$$=\binom{n}{n^*}\frac{M^2n^*+(1-m)^2\,(n-n^*)-4}{n(n-1)},$$

which is positive for all n large. Thus we get

$$\sum_{j<k}a(j,k)\leqslant\frac{n(n-1)}{n^*M^2+(1-m)^2\,(n-n^*)-4}=O(n).$$

Taking all $\varepsilon_j=1-m$, we get the lower bound from (9), $-(1-m)^{-2}\leqslant\sum_{j<k}a(j,k)$. Thus our statement follows and (6) holds.

(2) We prove now that the somewhat stronger condition (6′) implies D where

$$\sum_{1\leqslant j<k\leqslant n}|a(j,k)|\,\bar F_j(u_n(x))\,\bar F_k(u_n(x))=o(1) \tag{6′}$$

as $n\to\infty$. Let us denote $u_n=u_n(x)$ for any x fixed. Let $A=\{X_i\leqslant u_n,\,i\in I\}$ and $B=\{X_j\leqslant u_n,\,j\in J\}$ where I and J are disjoint subsets of $\{1,\ldots,n\}$, separated by m used in condition D. Let $S(I)=\sum_{j<k\in I}|a(j,k)|\,\bar F_j(u_n)\,\bar F_k(u_n)=o(1)$ for any $I\subset\{1,\ldots,n\}$. Then by (4)

$$\left| P(A \cap B) - \prod_{i \in I \cup J} F_i(u_n) \right| \leqslant S(I \cup J)$$

$$\left| P(A)\, P(B) - \prod_{i \in I \cup J} F_i(u_n) \right| \leqslant S(I) + S(J) + S(I)\, S(J),$$

which are all $o(1)$. Since this holds for any I, J, it implies D. Note that the separation of I of J is not used which shows the different nature of the conditions.

As in Lemma 1, we can show that (3) and (4) imply (6′). Note also that D is used to derive stronger results in the theory of extremes, as, e.g., the point process convergence of exceedances to a Poisson process.

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